Until we get to the magic, let's proceed as we did in Calculus. As we do this we'll be recalling facts and limit theorems/estimates from 3210-3220.

Theorem Let f be complex differentiable at $z_0 \in A$, $A \subseteq \mathbb{C}$ open. Then f is *continuous* at z_0.

Def f is continuous at $z_0 \in A$ means $\lim_{z \to z_0} f(z) = f(z_0)$

pf

$$f(z) = f(z_0) + (f(z) - f(z_0))$$

$$f(z) = f(z_0) + \frac{f(z)-f(z_0)}{z-z_0}(z-z_0) \qquad z \neq z_0.$$

$$\lim_{z\to z_0} \text{RHS} = \lim_{z\to z_0} f(z_0) + \lim(\;''\;) \qquad \text{limit thm for sums}$$

$$= f(z_0) + f'(z_0)\cdot 0 \qquad \text{limit thm for products} \qquad \square$$

Theorem Let $A \subseteq \mathbb{C}$ open, $f, g : A \to \mathbb{C}$ analytic, $c \in \mathbb{C}$. Then cf, $f+g$, fg are analytic on A. And the quotient $\frac{f}{g}$ is analytic in A intersect the complement of the zero set for g. Furthermore, for $z \in A$,

(i) $(cf)'(z) = cf'(z)$

(ii) $(f+g)'(z) = f'(z) + g'(z)$

(iii) $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

(iv) $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$ where $g(z) \neq 0$.

The proofs are just like in Calc 1. We can verify the product rule or the quotient rule, for example:

e.g. quotient rule. (Since $g(z_0) \neq 0$ & g cont, $g(z) \neq 0$ near enough z_0)

$$\left(\frac{f}{g}\right)'(z_0) = \lim_{z\to z_0} \frac{\frac{f(z)}{g(z)} - \frac{f(z_0)}{g(z_0)}}{z - z_0}$$

$$= \lim_{z\to z_0} \frac{f(z)g(z_0) - f(z_0)g(z)}{g(z)g(z_0)(z-z_0)}$$

rewrite numerator as
$f(z)g(z_0) - f(z_0)g(z_0)$
$+ f(z_0)g(z_0) - f(z_0)g(z)$

$$= \lim_{z\to z_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right)\frac{g(z_0)}{g(z)g(z_0)} + \frac{f(z_0)}{g(z)g(z_0)}\left(\frac{g(z_0)-g(z)}{z-z_0}\right) \to \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2} \quad \blacksquare$$

Some more computations that go just like in Calculus:

(i) if $f(z)$ is the constant function, its derivative is zero.

(ii) if $f(z) = z^n$, $n \in \mathbb{N}$, then $f'(z) = n\, z^{n-1}$

e.g. use binomial exp.

(iii) if $f(z) = z^n$, $n \in \mathbb{Z}$, then $f'(z) = n\, z^{n-1}$

use quotient rule for negative powers

(iv) every polynomial in z is analytic on $\mathbb{C}$, with the expected formula for its derivative.

sum & const. multiple rule.

(iv) if $f(z) = \dfrac{p(z)}{q(z)}$ is a rational function, i.e. a quotient of two polynomials, then $f(z)$ is analytic on the complement of the zero set for q.

quotient rule

The chain rule is also true - we'll prove this on Friday or next week, along with a discussion of the inverse function theorem. (The chain rule proof proceeds just like the precise proof for the 1-variable real chain rule that you discussed in 3210). In any case, if f is differentiable at z_0 and g is differentiable at $f(z_0)$ then $g \circ f$ is differentiable at z_0, and

$$(g \circ f)'(z_0) = g'(f(z_0))\, f'(z_0).$$

$\boxed{f(x+iy)=x}$

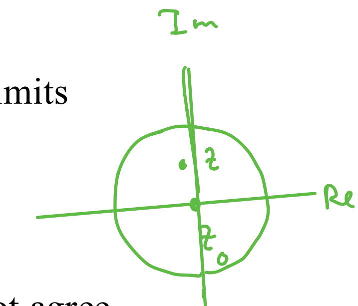
Example 3: Write $z = x + i\,y,\ y \in \mathbb{R}$. Then $\boxed{f(z) = \mathrm{Re}(z)} = x$ is NOT complex differentiable at any point of $\mathbb{C}$! (Even though the associated $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x, y) = (\mathrm{Re}\, f,\ \mathrm{Im}\, f) = (x,\ 0)$$

is Math 3220-differentiable, with differential (Jacobian) matrix

$$dF_{(x,\,y)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} !!! \quad 0$$

The way to check Example 3 at any point $z_0 = x_0 + i\,y_0$ is to evaluate the limits



$f(z) = \mathrm{Re}(z)$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{if it exists.}$$

$z \to z_0$ from the real and imaginary directions and see that these limits do not agree.

for full limit to exist, also directional limits from real & imag directions must exist & equal this full limit

Real dir.

$$\lim_{x+iy_0 \to x_0+iy_0} \frac{\mathrm{Re}(x+iy_0) - \mathrm{Re}(x_0+iy_0)}{x-x_0} = \lim_{x \to x_0} \frac{x-x_0}{x-x_0} = 1.$$

Im dir:

$$\lim_{x_0+iy \to x_0+iy_0} \frac{\mathrm{Re}(x_0+iy) - \mathrm{Re}(x_0+iy_0)}{i(y-y_0)} = \lim_{y \to y_0} \frac{0}{i(y-y_0)} = 0$$

So $\displaystyle\lim_{z \to z_0} \frac{\mathrm{Re}(z) - \mathrm{Re}(z_0)}{z - z_0}$ DNE!

$\frac{d}{dz} \mathrm{Re}\, z$ DNE!

In fact, being complex differentiable is very rare for a function $f: A \subseteq \mathbb{C} \to \mathbb{C}$, relatively speaking, even when $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are nice real-differentiable functions of x and y

<u>Theorem</u> Let $A \subseteq \mathbb{C}$ open, $f: A \to \mathbb{C}$, $z_0 \in A$. Write $f(z) = f(x + i\,y) = u(x, y) + i\,v(x, y)$, where $u(x, y) = \mathrm{Re}(f(x + i\,y)$, $v(x, y) = \mathrm{Im}(f(x + i\,y)$. Then if f is complex differentiable at $z_0 = x_0 + i\,y_0$ the following partial derivative equalities - known as the *Cauchy-Riemann equations* - must hold there:

$u_x := \frac{\partial u}{\partial x}$

CR
$$u_x(x_0, y_0) = v_y(x_0, y_0)$$
$$u_y(x_0, y_0) = -v_x(x_0, y_0).$$

$$\left(\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\right)$$

(The converse statement is almost true. The precise fact, which we'll discuss on ~~Friday~~ Wed., is that if $F: A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (u(x, y), v(x, y))$ is *Real differentiable* at (x_0, y_0) as you discussed in Math 3220, and if the CR equations hold at (x_0, y_0), *then* $\boxed{f(x + i\,y) = u(x, y) + i\,v(x, y)}$ is complex differentiable at $z_0 = x_0 + i\,y_0$. This is Theorem 1.5.8 in the text, which calls it the "Cauchy-Riemann Theorem". Geometrically, the CR Equations are saying that the differential map of F is given by a rotation-dilation matrix.)

proof: If $f'(z_0)$ $\exists$, then limit of diff qtnts from real & imag dirs $= f'(z_0)$. So

$$\lim_{x \to x_0}{}^{*} \frac{f(x+iy_0) - f(x_0+iy_0)}{x - x_0} = f'(z_0) = \lim_{y \to y_0} \frac{f(x_0+iy) - f(x_0+iy_0)}{i(y-y_0)}$$

$$\left(f(z) = u(x,y) + iv(x,y)\right)$$

$$= \frac{1}{i} \lim_{y \to y_0} \underbrace{\frac{f(x_0+iy) - f(x_0+iy_0)}{y-y_0}}_{:= f_y \ \left(\frac{\partial f}{\partial y}\right)}$$

$:= f_x$ $\left(\frac{\partial f}{\partial x}\right)$. Since $f = u + iv$ limit will equal

✓ $u_x(x_0,y_0) + iv_x(x_0,y_0)$

$$= \underbrace{\frac{1}{i}}_{-i}\left(u_y(x_0,y_0) + iv_y(x_0,y_0)\right)$$

$$= v_y(x_0,y_0) + i(-u_y(x_0,y_0))$$

check diff quot's: $\lim_{x \to x_0}{}^{*} \frac{u(x,y_0)+iv(x,y_0) - (u(x_0,y_0)+iv(x_0,y_0))}{x-x_0}$

$$= \lim_{x \to x_0} \frac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i \lim_{x \to x_0} \frac{v(x,y_0)-v(x_0,y_0)}{x-x_0}$$ ✓

Equate Re & Im parts, for $f'(z_0)$ $\exists$:

$u_x = v_y$ @ (x_0,y_0)

$v_x = -u_y$. CR

Name: Solutions

Math 4200 Quiz week 2 September 2, 2020

1a) For $w \in \mathbb{C} \setminus \{0\}$ express

$$w = |w|\, e^{i\, arg(w)}$$

where we choose

$$-\pi < arg(w) \le \pi.$$

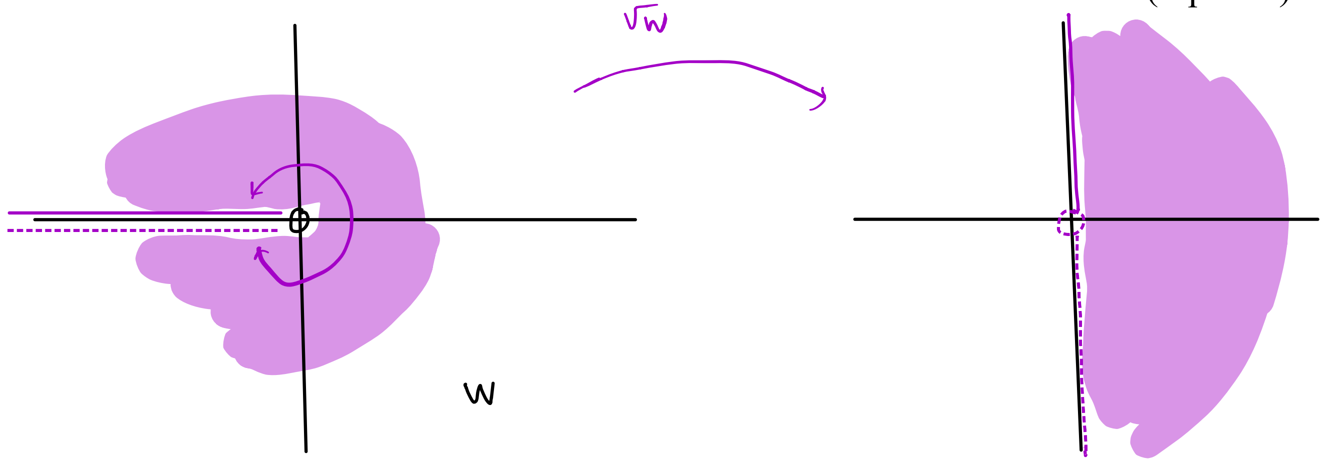
With this choice of argument we can define a branch of the square root function,

$$\sqrt{w} := |w|^{\frac{1}{2}}\, e^{i \frac{1}{2} arg\, w}.$$

Sketch the image of $\mathbb{C} \setminus \{0\}$ via this particular square root function.

$\sqrt{w}$

(5 points)



$-\pi < \arg w \le \pi \implies -\frac{\pi}{2} < \frac{1}{2}\arg w \le \frac{\pi}{2}$

Image is the right half plane (union pos imag. axis)

1b) Using the particular square root function above, for which $z \in \mathbb{C} \setminus \{0\}$ is it true that $\sqrt{z^2} = z$? Explain.

(5 points)

If $-\frac{\pi}{2} < \arg z \le \frac{\pi}{2}$, i.e. z in right half plane (union positive imaginary axis)

then since $\arg z^2 = 2\arg z$ deduce

$$-\pi < \arg z^2 \le \pi$$

so $\sqrt{z^2} = |z^2|^{1/2} e^{i\frac{\arg z^2}{2}} = |z|\, e^{i\frac{2\arg z}{2}} = z.$

If z is in the left half plane

then $\sqrt{z^2} \ne z$ because $\sqrt{z^2}$ is in the right half plane, by part (a).

Math 4200-001
Week 2 concepts and homework
1.5
Due ~~Wednesday September 9~~ at 5:00 p.m.
Friday Sept 11 @ 5:00 p.m.

1.5 1ad, 3b, 5c, 6c (in 5c and 6c describe the differential map as a rotation-dilation); 8, 9, 10, 11, 16, 18abc, 19.

w2.1a) Consider $f(z) = \frac{1}{z}$ and $z_0 = \frac{1+i}{2}$. Illustrate the rotation-dilation differential map for f at z_0 using rectangular coordinates. Precisely, Sketch a domain picture containing the point z_0 along with real and imaginary coordinate segments through that point having unit tangent vectors 1 and i. Sketch a range picture containing $f(1+i)$, the images of the coordinate segments with the corresponding image tangent vectors based at $f(z_0)$ - which should be rotated and dilated according to the argument and absolute value of $f'(z_0)$.

w2.1b) Repeat part (a), except using polar form. In other words, for $f\left(r\,\mathrm{e}^{i\,\theta}\right)$, $r_0 = \frac{1}{\sqrt{2}}$, $\theta_0 = \frac{\pi}{4}$, sketch r and θ coordinate segments through z_0 and their tangent vectors. In the range picture sketch the images of these coordinate segments and the corresponding rotated and dilated image tangent vectors.

In the problem above you are creating concrete realizations of the schematic pictures Figures 1.5.1 and 1.5.2 in the text.

Math 4200
Friday September 4

1.5 continued: The Cauchy-Riemann equations, chain rules, and the differential map

Announcements We'll talk about the Cauchy Riemann equations and Theorem in Wednesday's notes, and then discuss the chain rule and the differential map in today's notes. We'll briefly discuss the local inverse function theorem as well, leaving the in depth proofs of the Cauchy Riemann Theorem and local inverse function theorem until Wednesday next week. Each depends on key results from Math 3220, which we will state and articulate carefully to the present context. (The text omits both proofs.)

Postpone hw2 due date to next Friday @ 5:00 pm.
(because we didn't quite finish today's material)

Warm-up exercise

Our general discussion today will use the affine approximation characterization of complex differentiability. It is analogous to discussions you had in Math 3210-3220 when you discussed the "differential" or "differential matrix". $z = z_0 + h$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

Lemma: $f'(z_0)$ exists and has value c if and only if we have the affine approximation formula with error estimate:

$$f(z_0 + h) = f(z_0) + c\,h + \underbrace{h\,\varepsilon(h)}$$

where $\varepsilon(h) \to 0$ as $h \to 0$.

check: If $f'(z_0)$ exists then same as

$$\frac{f(z_0+h) - f(z_0)}{h} = f'(z_0) + \varepsilon(h) \qquad \text{where } \lim_{h\to 0} \varepsilon(h) \to 0$$

mult by h: $f(z_0+h) - f(z_0) = f'(z_0)h + h\varepsilon(h)$

add $f(z_0)$ to both sides ∎

Chain rules

1) Theorem (Chain rule for composition of analytic functions): If f is differentiable at z_0 and g is differentiable at $f(z_0)$ then $g \circ f$ is differentiable at z_0, and

$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$$

$$\mathbb{C} \xrightarrow{f} \mathbb{C}_k \xrightarrow{g} \mathbb{C}$$

$f(z_0) \to f(z_0+h)$

proof: We use the affine approximation formula for g at $f(z_0)$:

$$\underbrace{g(f(z_0+h)) - g(f(z_0))}_{h} = g'(f(z_0))(\underbrace{f(z_0+h) - f(z_0)}_{k}) + k\,\varepsilon(k)$$

g is diffble @ $f(z_0)$

for

$$k = f(z_0 + h) - f(z_0)$$

rewrite, divide by h:

$$\frac{g(f(z_0+h)) - g(f(z_0))}{h} = g'(f(z_0))\,\frac{f(z_0+h) - f(z_0)}{h} + \frac{k}{h}\varepsilon(k)$$

Take limits as $h \to 0$ and note that the last term $\to 0$ because $\frac{k}{h} \to f'(z_0)$ and $\varepsilon(k) \to 0$, since $k \to 0$ by the continuity of f.

f is diffble $\qquad 0$

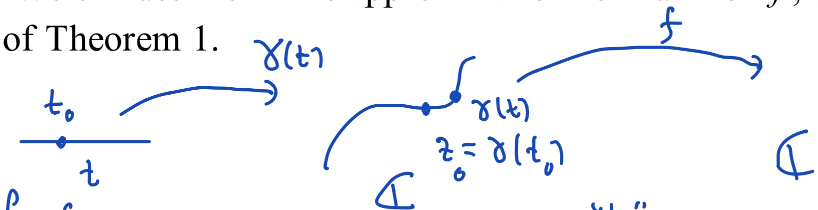
$$(g \circ f)'(z_0) = g'(f(z_0))f'(z_0).$$

QED

2) Theorem (Chain rule for curves) If f is differentiable at z_0 and $\gamma : I \subseteq \mathbb{R} \to \mathbb{C}$ is a parametric curve $\gamma(t) = x(t) + i\,y(t)$ such that $\gamma(t_0) = z_0$ and such that $\gamma'(t_0) = x'(t_0) + i\,y'(t_0)$ exists, then

$$(f\circ\gamma)'(t_0) = \underbrace{f'(\gamma(t_0))}\,\underbrace{\gamma'(t_0)}$$

proof We can use the affine approximation formula for f, at $\gamma(t_0)$, and mimic the proof of Theorem 1.



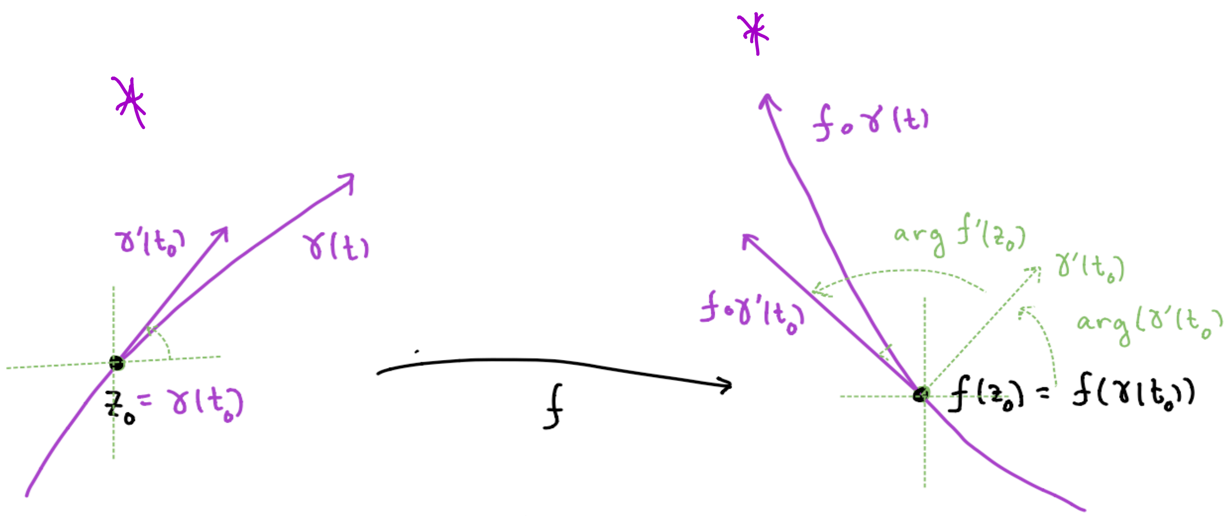
approx for f:

$$f(\gamma(t)) = f(\gamma(t_0)) + f'(\gamma(t_0))\overbrace{(\gamma(t)-\gamma(t_0))}^{\text{“}h\text{”}} + h\,\varepsilon(h) \qquad f \text{ diff'ble @ } z_0$$

$$\frac{f(\gamma(t))-f(\gamma(t_0))}{t-t_0} = f'(\gamma(t_0))\,\frac{\gamma(t)-\gamma(t_0)}{t-t_0} + \frac{h\,\varepsilon(h)}{t-t_0} \to \gamma'(t_0)\cdot 0 = 0$$

$\lim_{t\to t_0}$: $\boxed{(f\circ\gamma)'(t_0) = f'(\gamma(t_0))\,\gamma'(t_0) + 0}$

Domain-range geometry implied by the chain rule for curves. Consider the curve $\gamma(t)$ which has image in the domain of f, along with the curve $f\circ\gamma(t)$ which has image in the range of f. Let $f'(\gamma(t_0)) = r\,e^{i\theta}$. Then the image curve tangent vector is obtained by rotating the original curve tangent vector by r and scaling it by θ.



$$(f\circ\gamma)'(t) = \underbrace{f'(\gamma(t))}\,\gamma'(t)$$

@ t_0 $\quad \underbrace{|f'(z_0)|}_{\text{scaled by mod.}}\ \underbrace{e^{i\arg f'(z_0)}\,\gamma'(t_0)}_{\text{rotated by arg } f'(z_0)}$

Conformal transformations and differential map discussion:

(i) The precise definition of the *tangent space* at $z_0 \in \mathbb{C}$ is the set of all *tangent vectors* there, i.e. tangent vectors to curves passing through z_0:

$$T_{z_0}\mathbb{C} := \left\{ \gamma'(t_0) \;\middle|\; \gamma \text{ is differentiable at } t_0 \text{ and } \gamma(t_0) = z_0 \right\}$$

(ii) If $f(z)$ is a function from $\mathbb{C}$ to $\mathbb{C}$ that arises from a real-differentiable function $F : A \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, then the *differential of f at z_0* is defined by

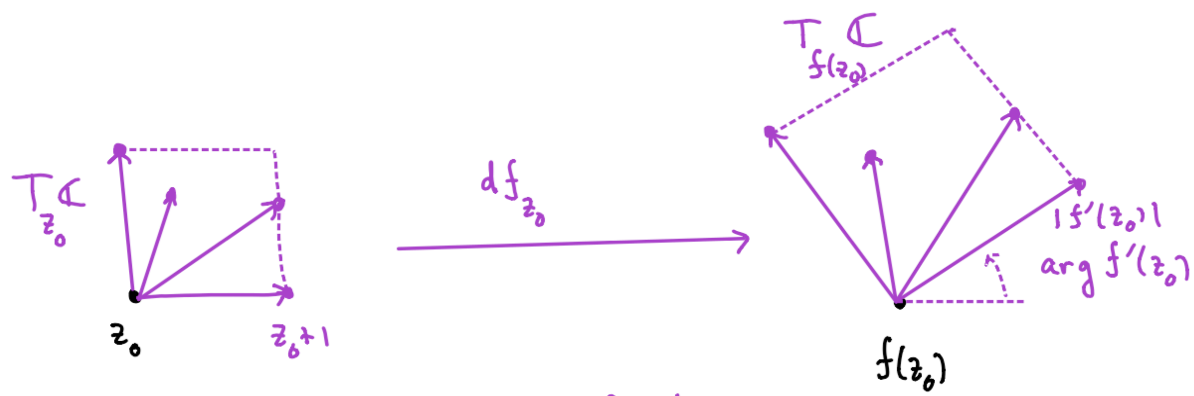
$$df_{z_0}\left(\gamma'(t_0)\right) := \left(f \circ \gamma\right)'(t_0).$$

$$df_{z_0} : T_{z_0}\mathbb{C} \to T_{f(z_0)}\mathbb{C}.$$

(iii) By the chain rule for curves, *if* $f(z)$ is complex differentiable at z_0, then

$$df_{z_0}\left(\gamma'(t_0)\right) := \left(f \circ \gamma\right)'(t_0) = f'(z_0)\gamma'(t_0)$$

Geometrically, this means that for complex differentiable functions f, the differential map is a linear transformation from $T_{z_0}\mathbb{C}$ to $T_{f(z_0)}\mathbb{C}$ which is a rotation-dilation.



$$df(\gamma'(t_0)) = f \circ \gamma'(t_0)$$
$$= f'(z_0)\gamma'(t_0)$$

(iv) A function $f : \mathbb{C} \to \mathbb{C}$ is called *conformal* at z_0 iff its differential transformation preserves angles between tangent vectors. Since rotation-dilations have this property, a function f which is complex differentiable at z_0, and for which $f'(z_0) \neq 0$, is conformal at z_0. (It turns out that if f is conformal at z_0 and also preserves orientations of pairs of tangent vectors, then f is complex differentiable at z_0.)

Illustration. Consider

$$\underline{f(z) = z^2,\ z_0 = 1 + i,}$$

$$f(z_0) = 2\,i,\ f'(z_0) = 2 + 2\,i = \underbrace{2\sqrt{2}}\,\underbrace{e^{i\frac{\pi}{4}}}$$

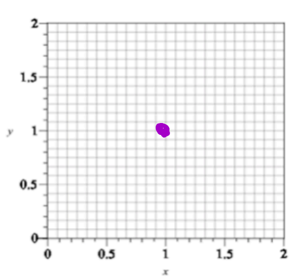
Below, are parts of a rectangular coordinate grid in the domain, and the image of that grid in the range space.

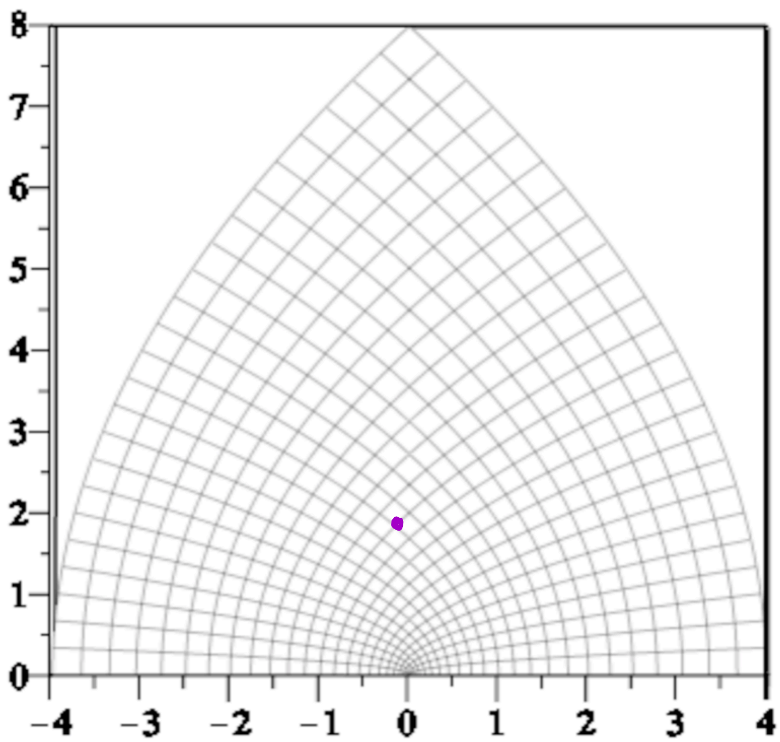
a) Why are the images of the real and imaginary grid lines also perpendicular?
b) Find the formula for the differential map

$$df_{z_0} : T_{z_0}\mathbb{C} \rightarrow T_{f(z_0)}\mathbb{C}$$

and illustrate the rotation dilation.

finish discussion on Wed. next week.





There is a local inverse function for analytic functions which we will prove on Wednesday next week using the local multivariable inverse function theorem you learned in Math 3220. I want to state it here, because it comes up in one of your homework problems for Wednesday. (You will only need to know the statement of the theorem, not its proof, for that problem.)

<u>Theorem</u> (Inverse function theorem) Let f be complex differentiable in a neighborhood of z_0, with $f'(z_0) \neq 0$ and $f'(z)$ continuous. Then there exist open sets U, V with $z_0 \in U$, $f(z_0) \in V$ such that $f: U \to V$ is a bijection and $f^{-1}: V \to U$ is also analytic. Furthermore

$$(f^{-1})'(f(z)) = \frac{1}{f(z)}$$

$\forall\, z \in U$.